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Implicit Application of Non-Reflective Boundary Conditions for Navier-Stokes Equations in Generalized Coordinates

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Background

• Internal unsteady flows sensitive to reflective waves at boundaries

• NRBC of Giles (1990) applicable for Euler eqs. with uniform far field flow

• NRBC of Poinsot and Lele (1992) developed for viscous flow using LODI in Cartesian Coordinates

• Kim and Lee (2000) extended NRBC of Poinsot and Lele to generalized coordinates, but their derivation has mistakes.

Objectives

- Extend the NRBC of Poinsot and Lele for Navier-Stokes equations to generalized coordinates
- Apply and test the NRBC

Governing Equations: 3D Compressible Navier-Stokes Equations:

$$\frac{\partial \mathbf{Q}'}{\partial t} + \frac{\partial \mathbf{E}'}{\partial \xi} + \frac{\partial \mathbf{F}'}{\partial \eta} + \frac{\partial \mathbf{G}'}{\partial \zeta} = \frac{1}{Re}\left(\frac{\partial \mathbf{E}'_{\mathbf{v}}}{\partial \xi} + \frac{\partial \mathbf{F}'_{\mathbf{v}}}{\partial \eta} + \frac{\partial \mathbf{G}'_{\mathbf{v}}}{\partial \zeta}\right) \tag{1}$$

$$\mathbf{Q}' = \frac{\mathbf{Q}}{J} \tag{2}$$

$$\mathbf{E}' = \frac{1}{J}(\xi_x \mathbf{E} + \xi_y \mathbf{F} + \xi_z \mathbf{G}) \tag{3}$$

$$\mathbf{F}' = \frac{1}{J}(\eta_x \mathbf{E} + \eta_y \mathbf{F} + \eta_z \mathbf{G}) \tag{4}$$

$$\mathbf{G}' = \frac{1}{J}(\zeta_x \mathbf{E} + \zeta_y \mathbf{F} + \zeta_z \mathbf{G}) \tag{5}$$

$$\mathbf{E}'_{\mathbf{v}} = \frac{1}{J}(\xi_x \mathbf{E}_{\mathbf{v}} + \xi_y \mathbf{F}_{\mathbf{v}} + \xi_z \mathbf{G}_{\mathbf{v}}) \tag{6}$$

$$\mathbf{F}'_{\mathbf{v}} = \frac{1}{J}(\eta_x \mathbf{E}_{\mathbf{v}} + \eta_y \mathbf{F}_{\mathbf{v}} + \eta_z \mathbf{G}_{\mathbf{v}}) \tag{7}$$

$$\mathbf{G}'_{\mathbf{v}} = \frac{1}{J}(\zeta_x \mathbf{E}_{\mathbf{v}} + \zeta_y \mathbf{F}_{\mathbf{v}} + \zeta_z \mathbf{G}_{\mathbf{v}}) \tag{8}$$

where the variable vector **Q**, and inviscid flux vectors **E**, **F**, and **G** are

$$\mathbf{Q} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho}\tilde{u} \\ \bar{\rho}\tilde{v} \\ \bar{\rho}\tilde{w} \\ \bar{\rho}\tilde{e} \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} \bar{\rho}\tilde{u} \\ \bar{\rho}\tilde{u}\tilde{u} + \tilde{p} \\ \bar{\rho}\tilde{u}\tilde{v} \\ \bar{\rho}\tilde{u}\tilde{w} \\ (\bar{\rho}\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix}, \mathbf{F} = \begin{pmatrix} \bar{\rho}\tilde{v} \\ \bar{\rho}\tilde{u}\tilde{v} \\ \bar{\rho}\tilde{v}\tilde{v} + \tilde{p} \\ \bar{\rho}\tilde{w}\tilde{v} \\ (\bar{\rho}\tilde{e} + \tilde{p})\tilde{v} \end{pmatrix}, \mathbf{G} = \begin{pmatrix} \bar{\rho}w \\ \bar{\rho}\tilde{u}\tilde{w} \\ \bar{\rho}\tilde{v}\tilde{w} \\ \bar{\rho}\tilde{w}\tilde{w} + \tilde{p} \\ (\bar{\rho}\tilde{e} + \tilde{p})\tilde{w} \end{pmatrix},$$

$$\mathbf{E_v} = \begin{pmatrix} 0 \\ \bar{\tau}_{xx} - \overline{\rho u'' u''} \\ \bar{\tau}_{xy} - \overline{\rho u'' v''} \\ \bar{\tau}_{xz} - \overline{\rho u'' w''} \\ Q_x \end{pmatrix}, \mathbf{F_v} = \begin{pmatrix} 0 \\ \bar{\tau}_{yx} - \overline{\rho v'' u''} \\ \bar{\tau}_{yy} - \overline{\rho v'' v''} \\ \bar{\tau}_{yz} - \overline{\rho v'' w''} \\ Q_y \end{pmatrix},$$

$$\mathbf{G_v} = \begin{pmatrix} 0 \\ \bar{\tau}_{zx} - \overline{\rho w'' u''} \\ \bar{\tau}_{zy} - \overline{\rho w'' v''} \\ \bar{\tau}_{zz} - \overline{\rho w'' w''} \\ Q_z \end{pmatrix}$$

$$\bar{\tau}_{ij} = -\frac{2}{3}\tilde{\mu}\frac{\partial \tilde{u}_k}{\partial x_k}\delta_{ij} + \tilde{\mu}(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}) \tag{9}$$

$$Q_i = \tilde{u}_j(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''}) - (\bar{q}_i + C_p\overline{\rho T'' u_i''}) \tag{10}$$

$$\bar{q}_i = -\frac{\tilde{\mu}}{(\gamma - 1)Pr}\frac{\partial a^2}{\partial x_i} \tag{11}$$

$$\bar{\rho}\tilde{e} = \frac{\tilde{p}}{(\gamma - 1)} + \frac{1}{2}\bar{\rho}(\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) + k \tag{12}$$

Baldwin-Lomax Turbulence Model used

Time Marching Scheme:

- Implicit Gauss-Seidel Relaxation, Dual Time Stepping

$$\frac{\partial \mathbf{Q}}{\partial t} = \frac{3Q^{n+1} - 4Q^{n} + Q^{n-1}}{2\Delta t} \tag{13}$$

$$\left[\left(\frac{1}{\Delta\tau} + \frac{1.5}{\Delta t}\right) I - \left(\frac{\partial R}{\partial Q}\right)^{n+1,m}\right] \delta Q^{n+1,m+1} = R^{n+1,m} - \frac{3Q^{n+1,m} - 4Q^{n} + Q^{n-1}}{2\Delta t} \tag{14}$$

$$R = -\frac{1}{V}\int_{s} [(\mathbf{E}' - \frac{1}{Re}\mathbf{E}'_v)\mathbf{i} + (\mathbf{F}' - \frac{1}{Re}\mathbf{F}'_v)\mathbf{j} + (\mathbf{G}' - \frac{1}{Re}\mathbf{G}'_v)\mathbf{k}] \cdot d\mathbf{s} \tag{15}$$

Characteristic Form of the Navier-Stokes Equations:

$$\mathbf{M}\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}\cdot\mathbf{M}\frac{\partial \mathbf{q}}{\partial \xi} + \mathbf{B}\cdot\mathbf{M}\frac{\partial \mathbf{q}}{\partial \eta} + \mathbf{C}\cdot\mathbf{M}\frac{\partial \mathbf{q}}{\partial \zeta} = \mathbf{R_v} \qquad (16)$$

$$\mathbf{A} = \frac{\partial \mathbf{E}'}{\partial \mathbf{Q}'}, \mathbf{B} = \frac{\partial \mathbf{F}'}{\partial \mathbf{Q}'}, \mathbf{C} = \frac{\partial \mathbf{G}'}{\partial \mathbf{Q}'}, \mathbf{M} = \frac{\partial \mathbf{Q}'}{\partial \mathbf{q}} \qquad (17)$$

$$\mathbf{q} = \frac{1}{J}\begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix} \qquad (18)$$

Equation (16) can be further expressed as:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{a}\frac{\partial \mathbf{q}}{\partial \xi} + \mathbf{b}\frac{\partial \mathbf{q}}{\partial \eta} + \mathbf{c}\frac{\partial \mathbf{q}}{\partial \zeta} = \mathbf{M^{-1}R_v} \tag{19}$$

$$\mathbf{a} = \mathbf{M^{-1}AM}, \mathbf{b} = \mathbf{M^{-1}BM}, \mathbf{c} = \mathbf{M^{-1}CM} \tag{20}$$

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ have the same eigenvalues as Jacobian matrix $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

$$\mathbf{a} = \mathbf{P\Lambda P}^{-1} \tag{21}$$

$$\mathbf{\Lambda} = \begin{pmatrix} U & 0 & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & U+C & 0 \\ 0 & 0 & 0 & 0 & U-C \end{pmatrix} \tag{22}$$

The Navier-Stokes equation, then can be expressed as:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}\frac{\partial \mathbf{q}}{\partial \xi} + \mathbf{b}\frac{\partial \mathbf{q}}{\partial \eta} + \mathbf{c}\frac{\partial \mathbf{q}}{\partial \zeta} = \mathbf{M^{-1}R_v} \tag{23}$$

or

$$\mathbf{P^{-1}}\frac{\partial \mathbf{q}}{\partial t} + \mathbf{\Lambda}\mathbf{P}^{-1}\frac{\partial \mathbf{q}}{\partial \xi} + \mathbf{P^{-1}b}\frac{\partial \mathbf{q}}{\partial \eta} + \mathbf{P^{-1}c}\frac{\partial \mathbf{q}}{\partial \zeta} = \mathbf{P^{-1}M^{-1}R_v} \tag{24}$$

$\mathbf{P^{-1}}$ can not be absorbed into $\frac{\partial \mathbf{q}}{\partial t}$ and $\frac{\partial \mathbf{q}}{\partial x}$. It is incorrect to express NS eqs. as:

$$\frac{\partial \mathbf{R}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{R}}{\partial \xi} + = \mathbf{P^{-1}} S_v^* \tag{25}$$

Characteristic form of the Navier-Stokes equations in ξ direction.

$$\mathcal{L} = \mathbf{\Lambda} \mathbf{P^{-1}} \frac{\partial \mathbf{q}}{\partial \xi} \tag{26}$$

$$\mathbf{P^{-1}}\frac{\partial \mathbf{q}}{\partial t} + \mathcal{L} + \mathbf{P^{-1}}\mathbf{b}\frac{\partial \mathbf{q}}{\partial \eta} + \mathbf{P^{-1}}\mathbf{c}\frac{\partial \mathbf{q}}{\partial \zeta} = \mathbf{P^{-1}}\mathbf{M^{-1}}\mathbf{R_v} \tag{27}$$

$\mathcal{L}$: the amplitude of the characteristic waves

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \mathcal{L}_5 \end{pmatrix} = \begin{pmatrix} U[\tilde{\xi}_x \frac{\partial}{\partial \xi}(\frac{\rho}{J}) + \tilde{\xi}_z \frac{\partial}{\partial \xi}(\frac{v}{J}) - \tilde{\xi}_y \frac{\partial}{\partial \xi}(\frac{w}{J}) - \frac{\tilde{\xi}_x}{c^2} \frac{\partial}{\partial \xi}(\frac{p}{J})] \\ U[\tilde{\xi}_y \frac{\partial}{\partial \xi}(\frac{\rho}{J}) - \tilde{\xi}_z \frac{\partial}{\partial \xi}(\frac{u}{J}) + \tilde{\xi}_x \frac{\partial}{\partial \xi}(\frac{w}{J}) - \frac{\tilde{\xi}_y}{c^2} \frac{\partial}{\partial \xi}(\frac{p}{J})] \\ U[\tilde{\xi}_z \frac{\partial}{\partial \xi}(\frac{\rho}{J}) + \tilde{\xi}_y \frac{\partial}{\partial \xi}(\frac{u}{J}) - \tilde{\xi}_x \frac{\partial}{\partial \xi}(\frac{v}{J}) - \frac{\tilde{\xi}_z}{c^2} \frac{\partial}{\partial \xi}(\frac{p}{J})] \\ (U + C)[\frac{\tilde{\xi}_x}{\sqrt{2}} \frac{\partial}{\partial \xi}(\frac{u}{J}) + \frac{\tilde{\xi}_y}{\sqrt{2}} \frac{\partial}{\partial \xi}(\frac{v}{J}) + \frac{\tilde{\xi}_z}{\sqrt{2}} \frac{\partial}{\partial \xi}(\frac{w}{J}) + \beta \frac{\partial}{\partial \xi}(\frac{p}{J})] \\ (U - C)[-\frac{\tilde{\xi}_x}{\sqrt{2}} \frac{\partial}{\partial \xi}(\frac{u}{J}) - \frac{\tilde{\xi}_y}{\sqrt{2}} \frac{\partial}{\partial \xi}(\frac{v}{J}) - \frac{\tilde{\xi}_z}{\sqrt{2}} \frac{\partial}{\partial \xi}(\frac{w}{J}) + \beta \frac{\partial}{\partial \xi}(\frac{p}{J})] \end{pmatrix} \tag{28}$$

Express the Characteristic form of the Navier-Stokes equations in conservative variables:

$$\frac{\partial \mathbf{Q}'}{\partial t} + \mathcal{D} + \frac{\partial \mathbf{F}'}{\partial \eta} + \frac{\partial \mathbf{G}'}{\partial \zeta} = \frac{1}{Re}\left(\frac{\partial \mathbf{E}'_{\mathbf{v}}}{\partial \xi} + \frac{\partial \mathbf{F}'_{\mathbf{v}}}{\partial \eta} + \frac{\partial \mathbf{G}'_{\mathbf{v}}}{\partial \zeta}\right) \tag{29}$$

$$\mathcal{D} = \mathbf{M}\mathbf{P}\mathcal{L} \tag{30}$$

Local One-Dimensional Inviscid (LODI):

$$\frac{\partial \mathbf{Q}'}{\partial t} + \mathcal{D} = 0 \tag{31}$$

Implicit Implementation of the NRBC

$$\left[\left(\frac{1}{\Delta\tau} + \frac{1.5}{\Delta t}\right) I - \left(\frac{\partial R_{bc}}{\partial Q}\right)^{n+1,m} + \left(\frac{\partial \mathcal{D}}{\partial Q}\right)^{n+1,m}\right] \delta Q^{n+1,m+1}$$

$$= R_{bc}^{n+1,m} - \mathcal{D}^{n+1,m} - \frac{3Q^{n+1,m} - 4Q^{n} + Q^{n-1}}{2\Delta t} \tag{32}$$

$$R_{bc} = -\frac{1}{V}\int_s [(-\frac{1}{Re}\mathbf{E}'_v)\mathbf{i} + (\mathbf{F}' - \frac{1}{Re}\mathbf{F}'_v)\mathbf{j} + (\mathbf{G}' - \frac{1}{Re}\mathbf{G}'_v)\mathbf{k}]\cdot d\mathbf{s} \tag{33}$$

Supersonic outflow boundary conditions

- All eigenvalues positive
- The complete characteristic N-S eqs (Eq. 29) solved.
- $\mathcal{L}$ evaluated using first or second order upwind differencing.

Subsonic outflow boundary conditions

- Eigenvalue $U - C$ negative
- The characteristic N-S eqs (Eq. 29) solved.
- Soft boundary condition

$$\mathcal{L}_5 = \mathcal{K}(p - p_e),\ \mathcal{K} = \sigma|1 - \mathcal{M}^2|/(\sqrt{2}J\rho L) \tag{34}$$

If $\mathcal{L}_5 = 0$, 'perfect' non-reflective boundary condition.

- Three zero gradient for viscous terms:

Subsonic inflow boundary conditions

• $\mathcal{L}_{1-4}$ enter the domain, $\mathcal{L}_5$ leave the domain.

• Four conditions are given, the energy eq. solved with $\mathcal{L}_5$ using one-side differencing.

• $\mathcal{L}_1$ - $\mathcal{L}_4$ obtained using LODI:

$$\mathcal{L}_1 = -\tilde{\xi}_x \frac{\rho}{\sqrt{2}c}(\mathcal{L}_4+\mathcal{L}_5), \;\; \mathcal{L}_2 = -\tilde{\xi}_y \frac{\rho}{\sqrt{2}c}(\mathcal{L}_4+\mathcal{L}_5), \;\; \mathcal{L}_3 = -\tilde{\xi}_z \frac{\rho}{\sqrt{2}c}(\mathcal{L}_4+\mathcal{L}_5), \;\; \mathcal{L}_4 \tag{35}$$

Adiabatic wall boundary conditions

• $u_o = -u_i$, $v_o = -v_i$, $w_o = -w_i$. $\frac{\partial T}{\partial \eta} = 0$.

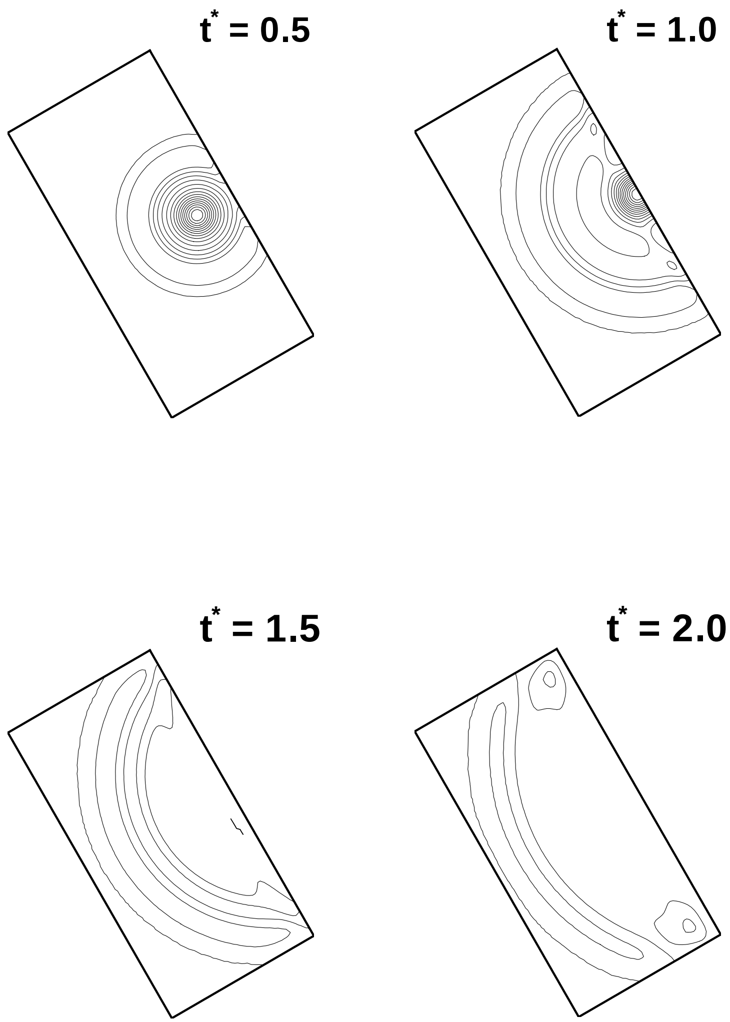
• total energy ρe_o is solved by energy equation

• Cross η boundary

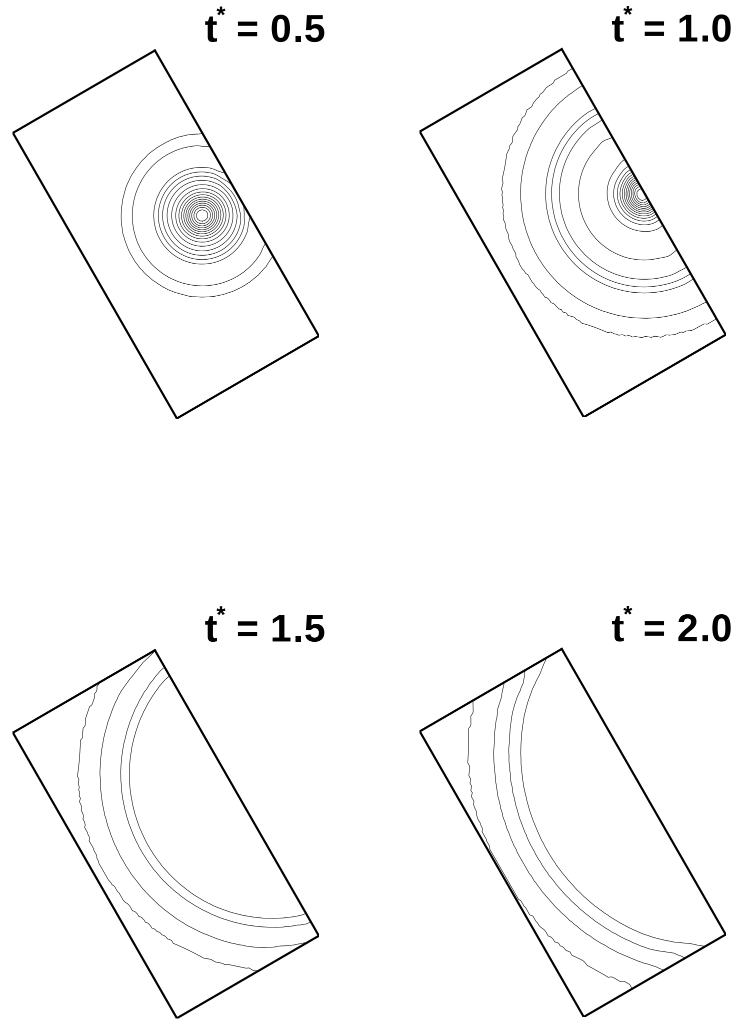
$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \mathcal{L}_5 \end{pmatrix} = \begin{pmatrix} V[\tilde{\eta}_x \frac{\partial}{\partial \eta}(\frac{\rho}{J}) + \tilde{\eta}_z \frac{\partial}{\partial \eta}(\frac{v}{J}) - \tilde{\eta}_y \frac{\partial}{\partial \eta}(\frac{w}{J}) - \frac{\tilde{\eta}_x}{c^2} \frac{\partial}{\partial \eta}(\frac{p}{J})] \\ V[\tilde{\eta}_y \frac{\partial}{\partial \eta}(\frac{\rho}{J}) - \tilde{\eta}_z \frac{\partial}{\partial \eta}(\frac{u}{J}) + \tilde{\eta}_x \frac{\partial}{\partial \eta}(\frac{w}{J}) - \frac{\tilde{\eta}_y}{c^2} \frac{\partial}{\partial \eta}(\frac{p}{J})] \\ V[\tilde{\eta}_z \frac{\partial}{\partial \eta}(\frac{\rho}{J}) + \tilde{\eta}_y \frac{\partial}{\partial \eta}(\frac{u}{J}) - \tilde{\eta}_x \frac{\partial}{\partial \eta}(\frac{v}{J}) - \frac{\tilde{\eta}_z}{c^2} \frac{\partial}{\partial \eta}(\frac{p}{J})] \\ (V+C)[\frac{\tilde{\eta}_x}{\sqrt{2}} \frac{\partial}{\partial \eta}(\frac{u}{J}) + \frac{\tilde{\eta}_y}{\sqrt{2}} \frac{\partial}{\partial \eta}(\frac{v}{J}) + \frac{\tilde{\eta}_z}{\sqrt{2}} \frac{\partial}{\partial \eta}(\frac{w}{J}) + \beta \frac{\partial}{\partial \eta}(\frac{p}{J})] \\ (V-C)[-\frac{\tilde{\eta}_x}{\sqrt{2}} \frac{\partial}{\partial \eta}(\frac{u}{J}) - \frac{\tilde{\eta}_y}{\sqrt{2}} \frac{\partial}{\partial \eta}(\frac{v}{J}) - \frac{\tilde{\eta}_z}{\sqrt{2}} \frac{\partial}{\partial \eta}(\frac{w}{J}) + \beta \frac{\partial}{\partial \eta}(\frac{p}{J})] \end{pmatrix} \tag{36}$$

- $\mathcal{L}_1$ - $\mathcal{L}_3 = 0$, $\mathcal{L}_4 = \mathcal{L}_5$ from LODI

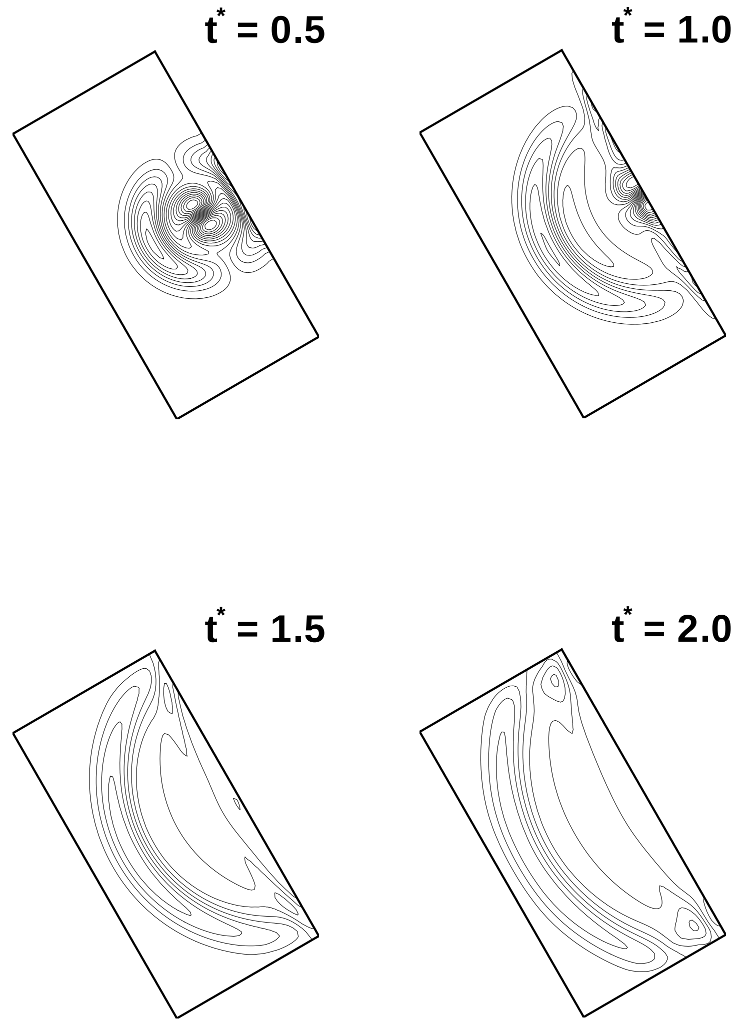
Results: Density, Subsonic Moving Vortex with Imposed p at exit



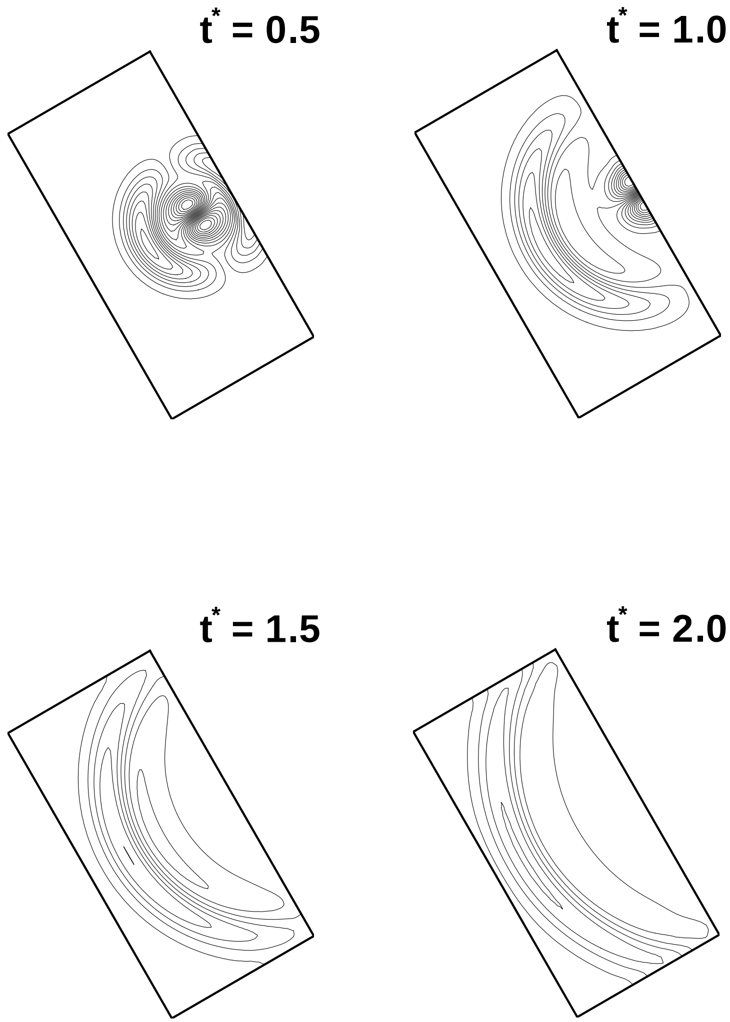
Results: Density, Subsonic Moving Vortex with NRBC

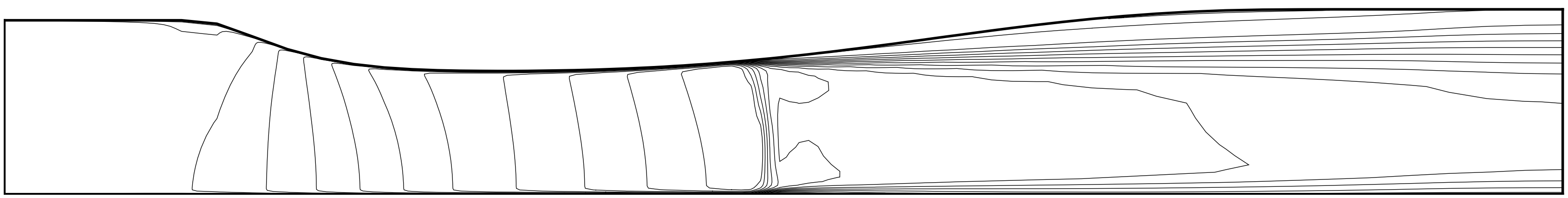


$(u^\tau - u^\tau{}_\infty)/u^\tau{}_\infty$ of Subsonic Moving Vortex with Imposed p at exit

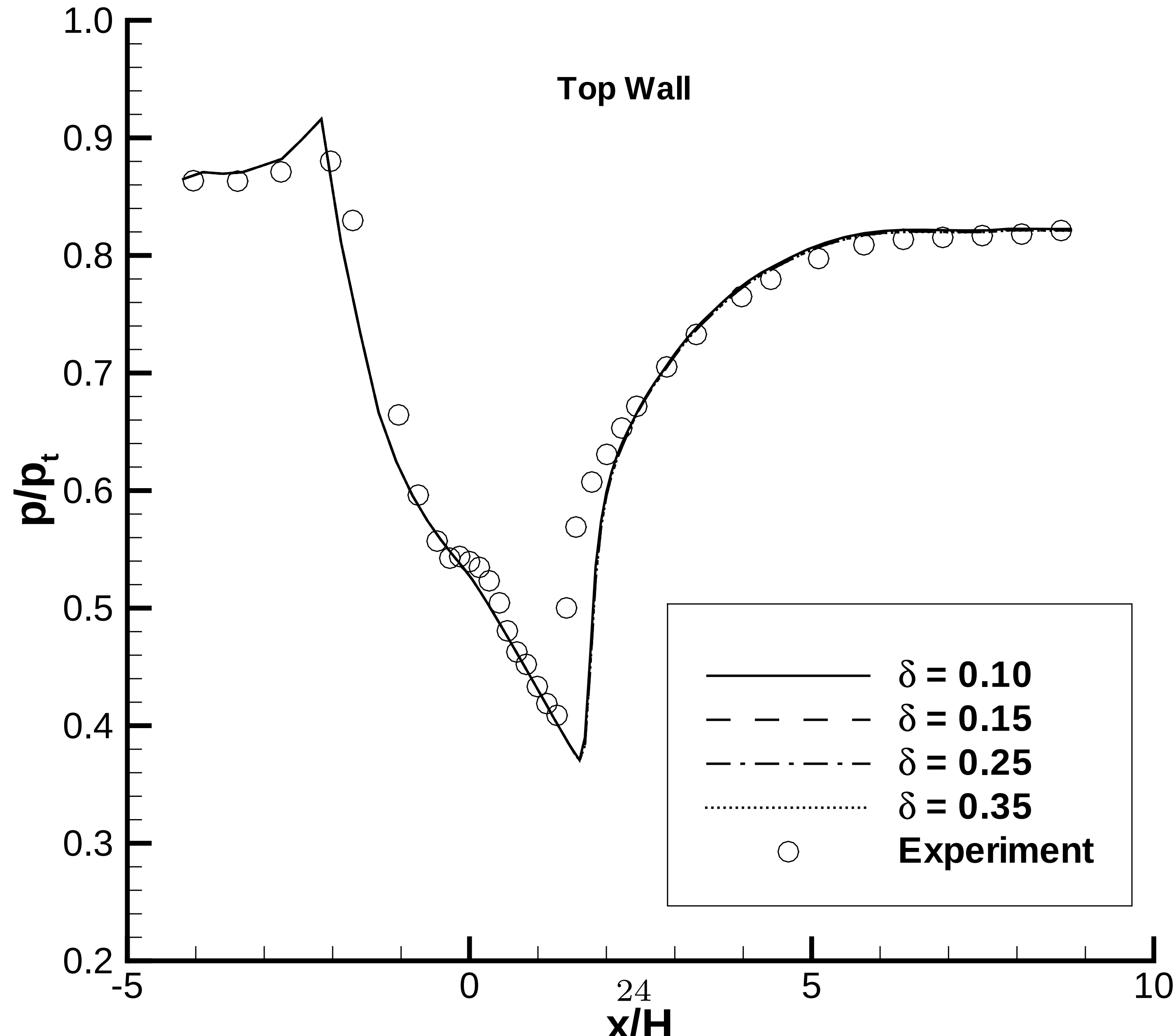


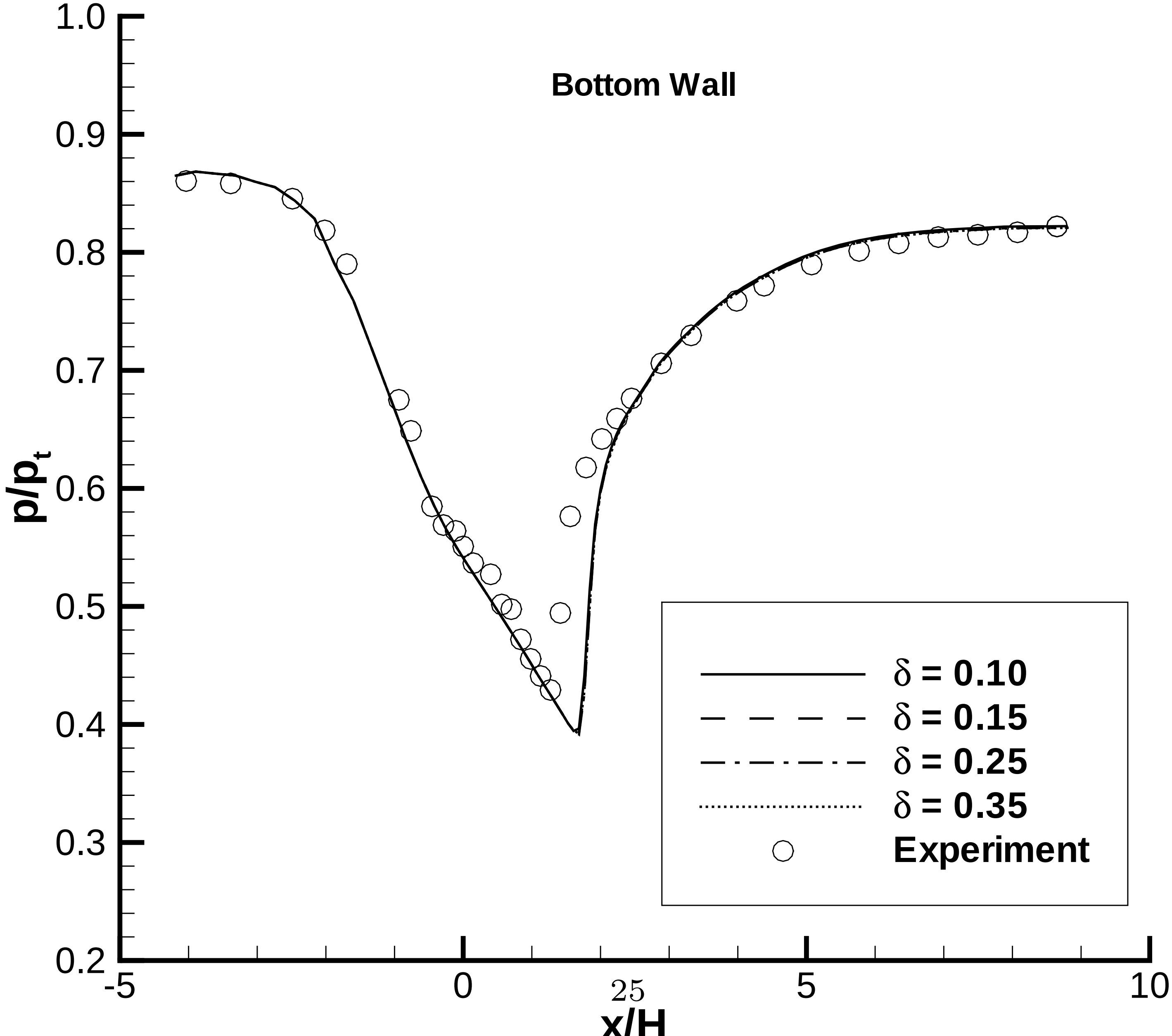
$(u^\tau - {u^\tau}_\infty)/{u^\tau}_\infty$ of Subsonic Moving Vortex with NRBC

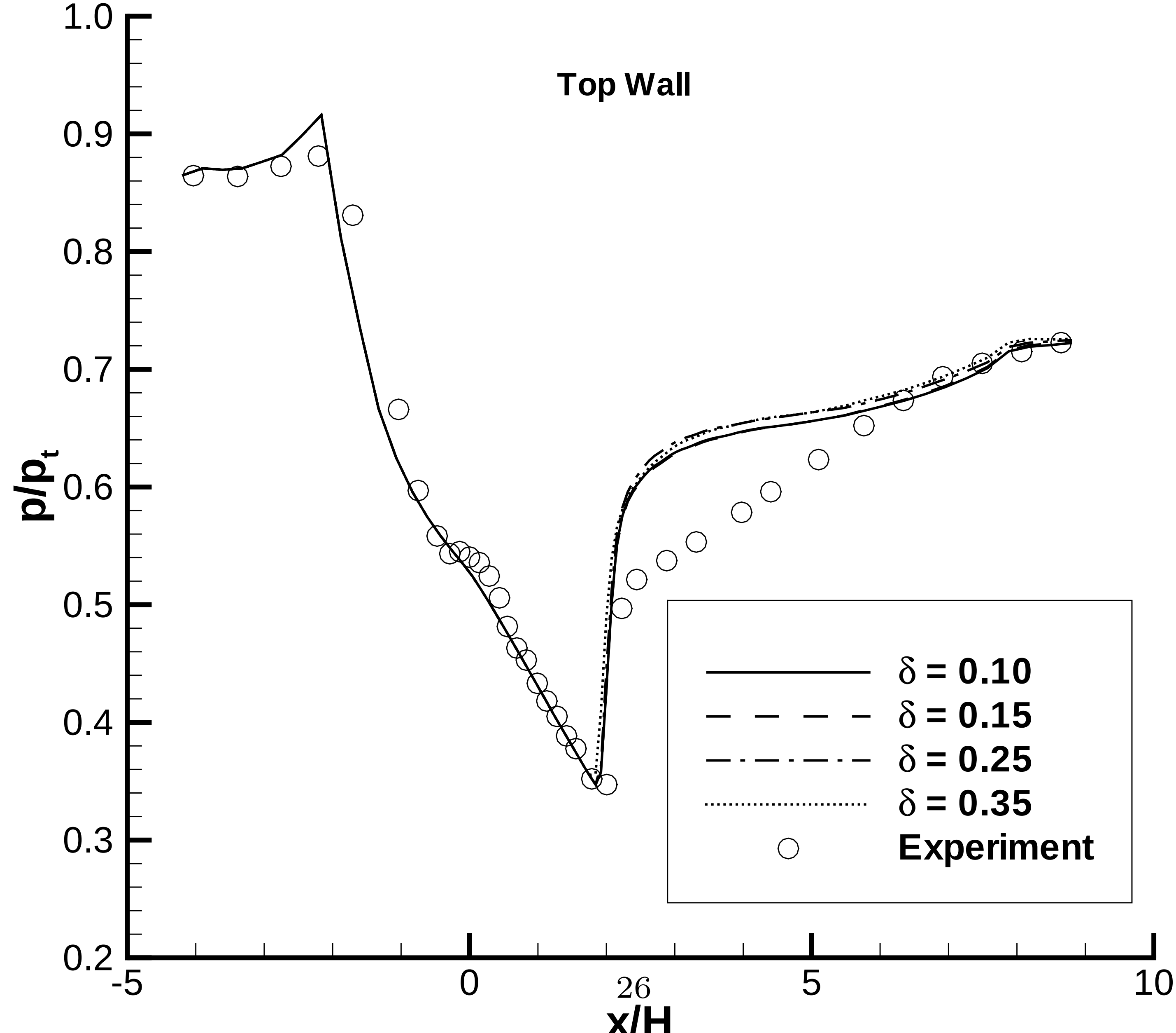




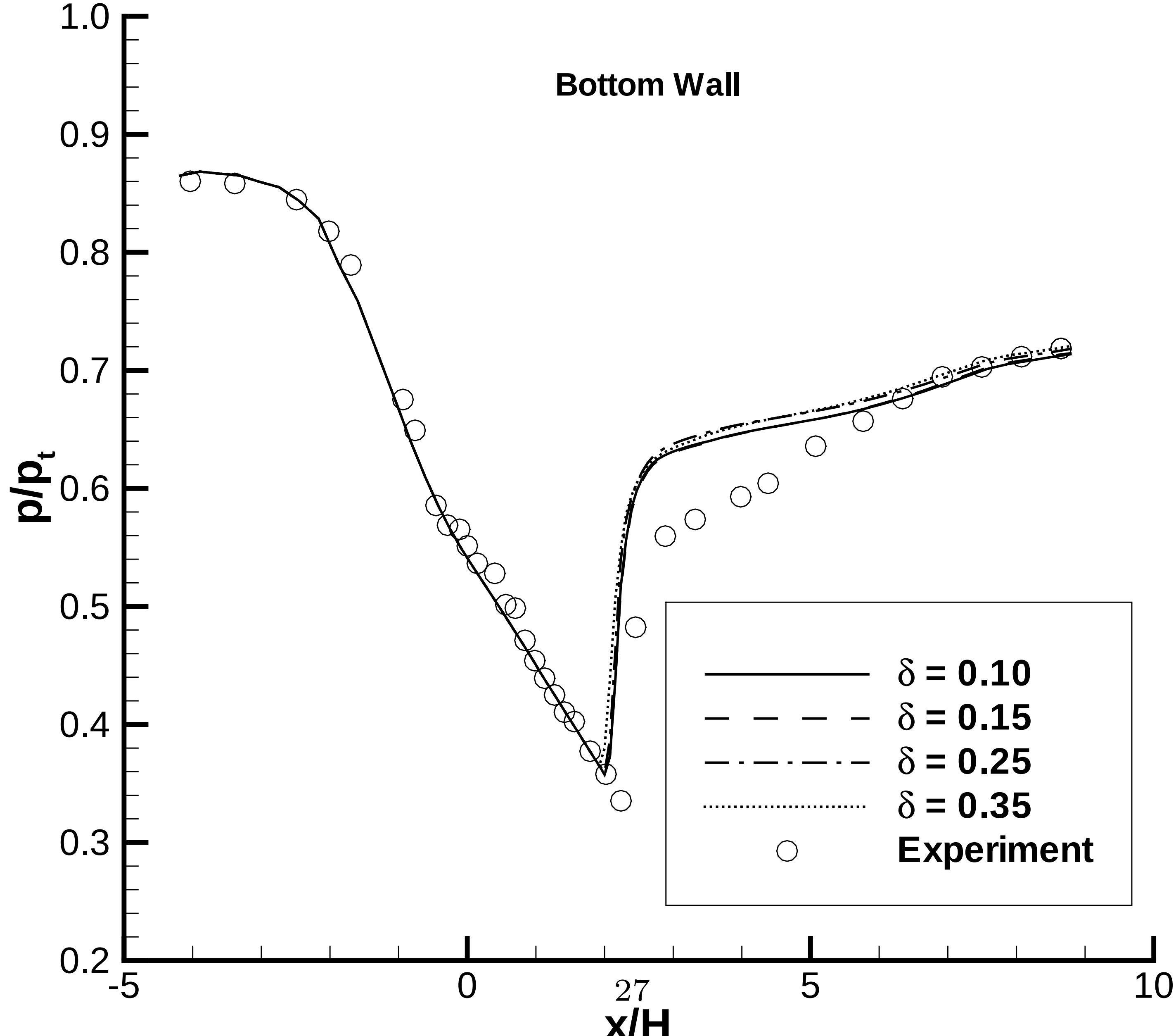
Steady state *Mach* number contours of the inlet diffuser with NRBC.


Top Wall
p/p_t
x/H
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-5
0
5
10
δ = 0.10
δ = 0.15
δ = 0.25
δ = 0.35
Experiment

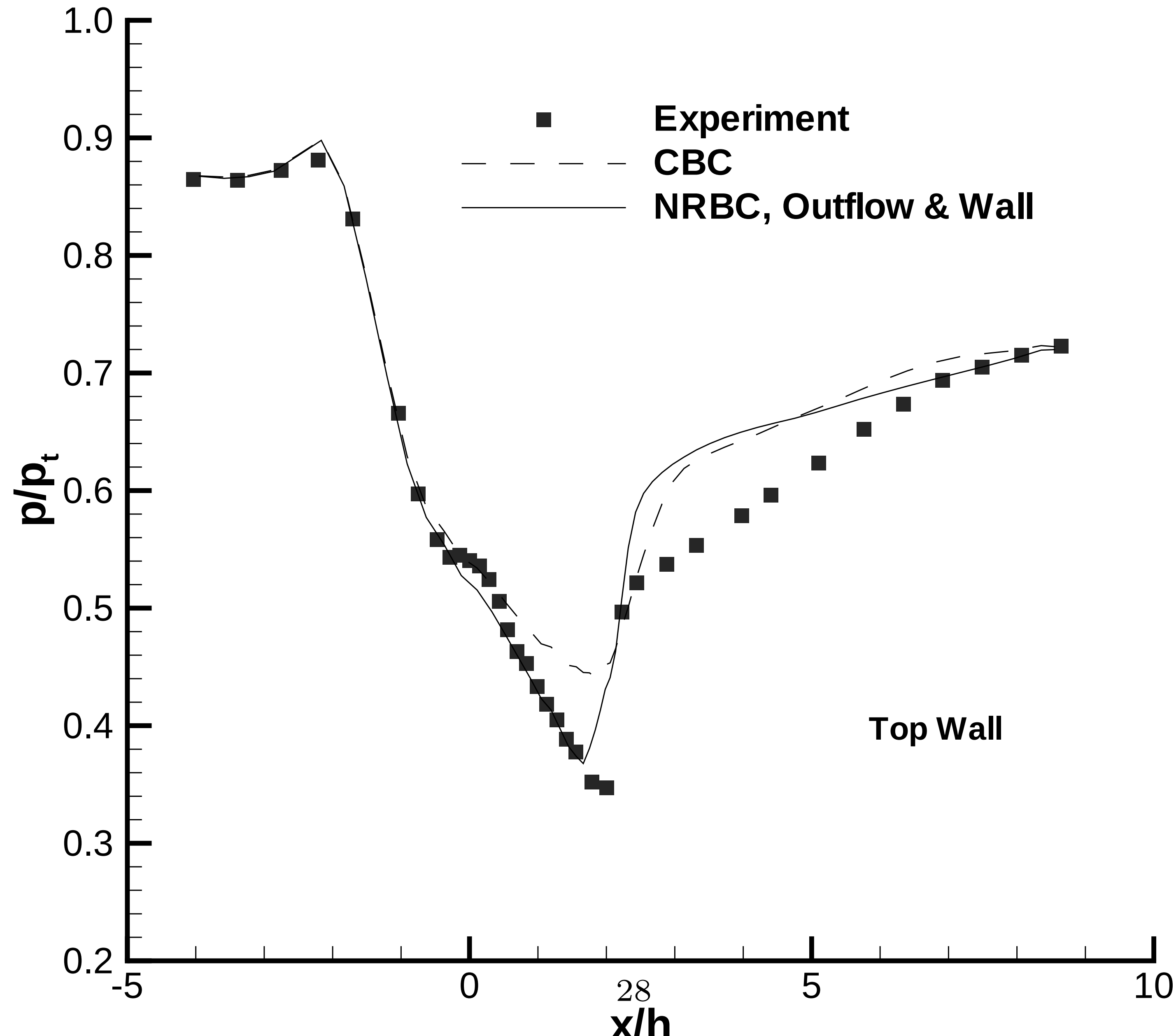

Bottom Wall
p/p$_t$
x/H
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-5
0
5
10
δ = 0.10
δ = 0.15
δ = 0.25
δ = 0.35
Experiment

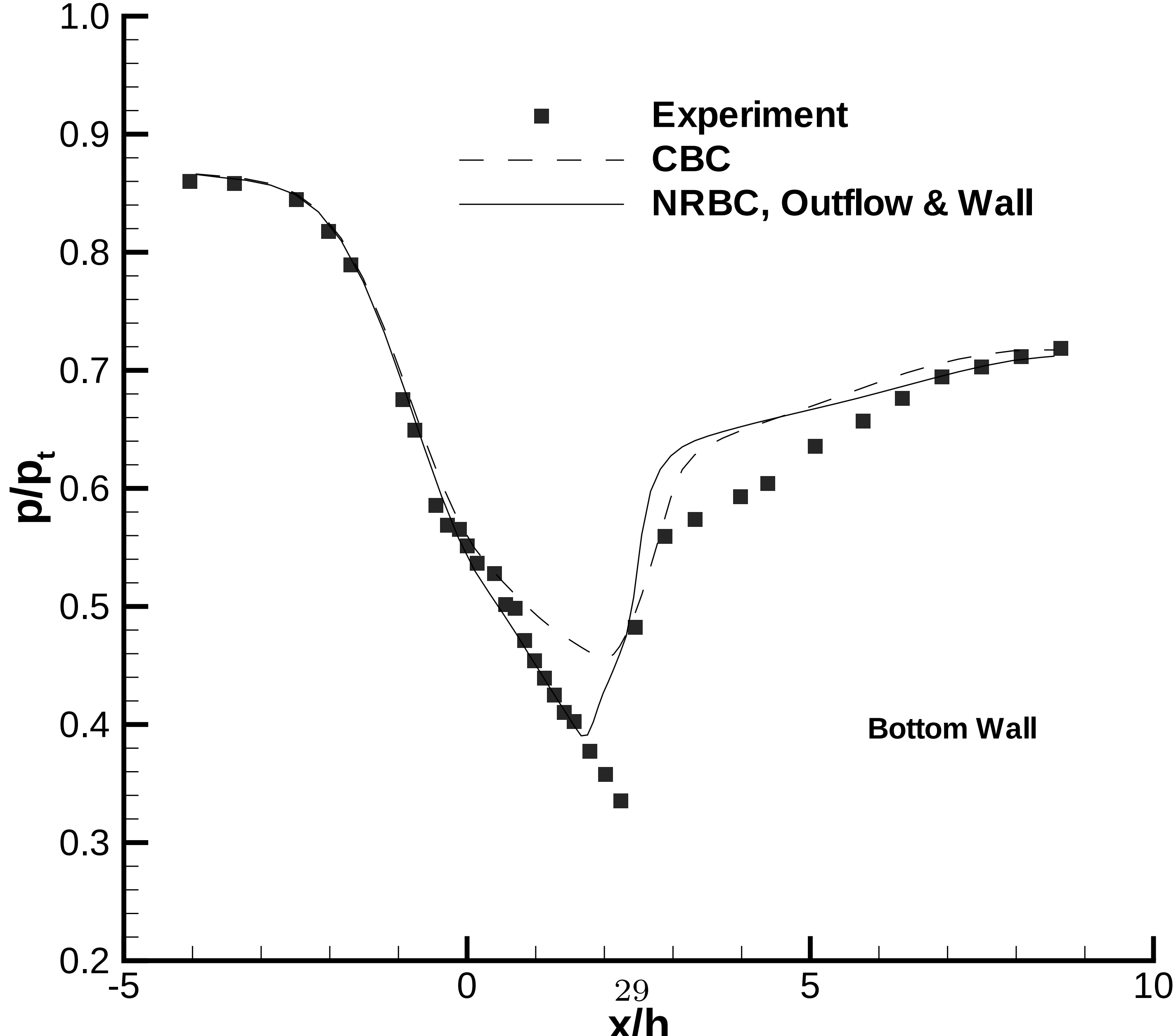


Top Wall
p/p_t
x/H
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-5
0
5
10
δ = 0.10
δ = 0.15
δ = 0.25
δ = 0.35
Experiment

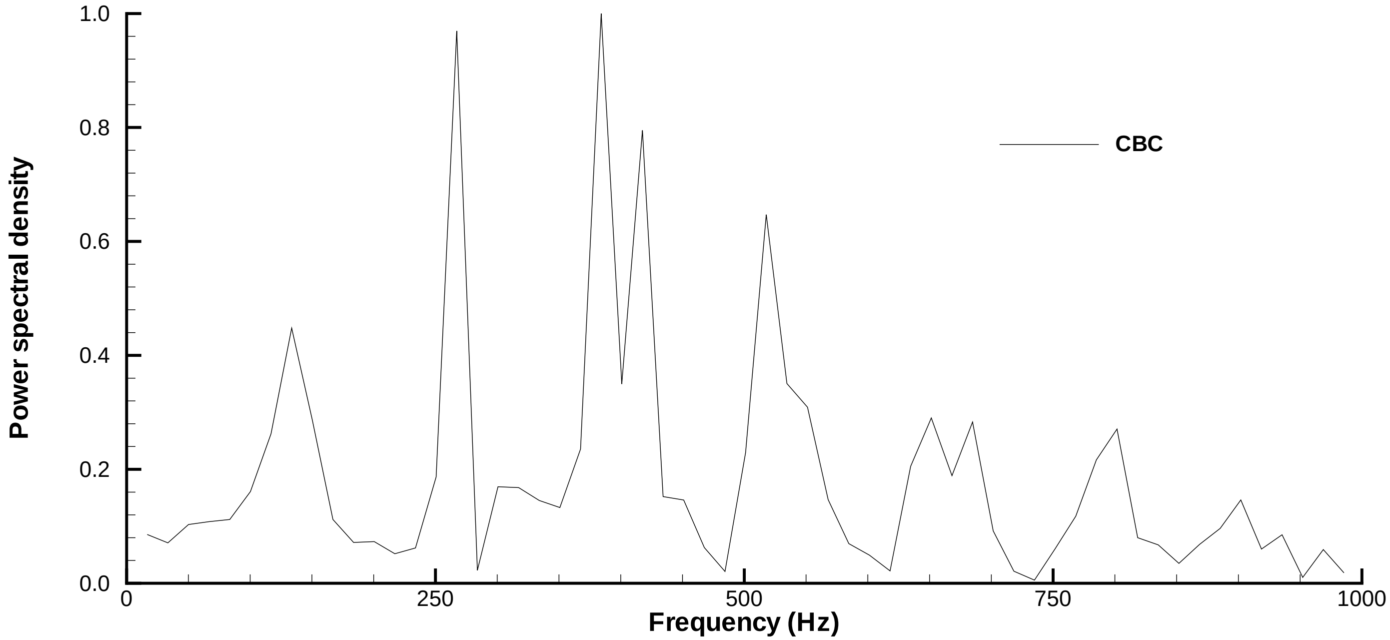


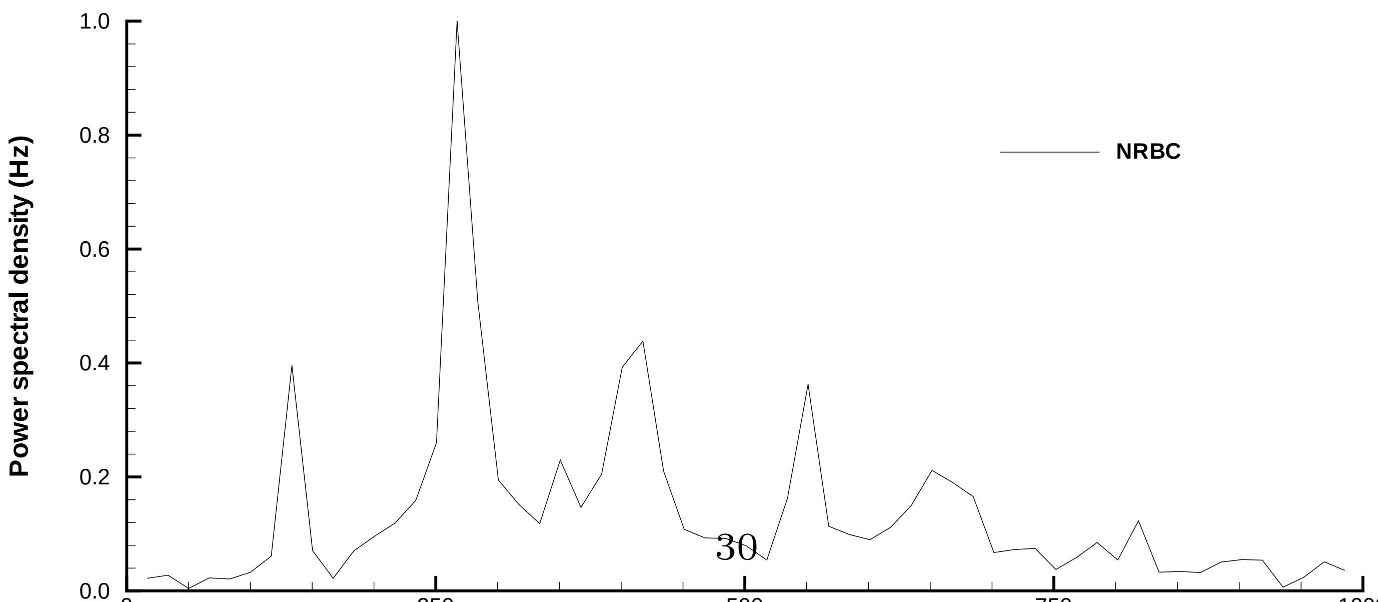
Bottom Wall
p/p_t
x/H
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-5
0
5
10
δ = 0.10
δ = 0.15
δ = 0.25
δ = 0.35
Experiment


Experiment
CBC
NRBC, Outflow & Wall
Top Wall
p/p_t
x/h
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-5
0
5
10



Experiment
CBC
NRBC, Outflow & Wall
Bottom Wall
p/p_t
x/h
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-5
0
5
10


Power spectral density
1.0
0.8
0.6
0.4
0.2
0.0
0
250
500
750
1000
Frequency (Hz)
CBC


Power spectral density (Hz)
1.0
0.8
0.6
0.4
0.2
0.0
NRBC

Conclusions

- The NRBC of Poinsot and Lele for 3D Navier-Stokes equations are extended to generalized coordinates with detailed formulations
- The NRBC is implemented implicitly and is coupled with the inner domain solver by Gauss-Seidel Iteration
- For a unsteady subsonic vortex propagating flow, the NRBC avoid flow distortion at boundaries caused the BCs specifying pressure.
- For a steady state transonic inlet-diffuser, the NRBC is not essential. The reflective waves are diffused when the solutions are converged.
- For an unsteady transonic inlet-diffuser, the NRBC is essential.
- For the exit BC with imposed pressure, the shock wave is largely oscillated by reflective waves.

• When NRBC is used at exit, the shock oscillation dramatically reduced, the computed time averaged pressure distributions and frequency agree much better with the experiment

Acknowledgment

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